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BEAM RESTRAINTS PROVIDED BY WALLS WITH OPENINGS

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BEAM RESTRAINTS PROVIDED BY WALLS WITH OPENINGS

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SYNOPSIS

No generally accepted design criteria are available for determining the amount of restraint provided by a reinforced concrete wall to a beam framing into it. By means of graphs this paper attempts to provide this information in a convenient form. The results of an analytical solution for a plate, considered to be equivalent to a wall, to which a local edge moment is applied are reported. The effective stiffness of the plate is converted into an equivalent beam stiffness. Then, based on experimental evidence, the reduction in restraint due to certain type of rectangular openings is indicated. These results are shown on graphs. These graphs enable one to determine very simply the effective stiffness of a plate or a wall without openings, or with openings of different heights, but of width equal to the height of the wall, in terms of an equivalent beam. The proposed method of analysis is based on the assumption of the elastic behavior of material.

INTRODUCTION

An element of a very common type of construction—the reinforced concrete bearing wall—has received relatively little attention by the analysts. No generally accepted design criteria are available for determining the amount of restraint provided by a wall to a beam framing into it. However, such information is necessary for the proper design of the reinforcement of a beam as well as a wall.

The stated problem is solved by idealizing the actual situation. First, it is assumed that the wall behaves elastically, and that the material from which it is made is isotropic. This assumption, although somewhat at variance with the actual facts, is nevertheless commonly used in the analysis and design of reinforced concrete flexural members. Second, since the axial stresses in a bearing wall are very small, their effect is neglected in determining the flexural stiffness of the wall. Based on these assumptions, the determination of the effective stiffness of a wall reduces to a problem of an elastic plate subjected to a locally applied moment. In the following, an analytical solution of this problem is discussed. The solution established is applicable to a solid, rectangular, simply supported plate with a local moment applied parallel to one pair

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of the plate's edges. The solution shows that for the purpose of determining the effective stiffness of a solid plate to a locally applied moment, it is possible to replace the wall with an equivalent beam.

When the plate is weakened by openings, such as window and door openings in the case of a bearing wall, the determination of the stiffness of the weakened plate becomes very involved mathematically. An analytical solution of this problem was not attempted. Instead, experimental investigations were carried out on a steel plate with various size rectangular openings. The height of these openings was varied, however their width in all cases was equal to the height of the plate. The results of this investigation provide experimentally determined factors which in per cent give the amount of restraint given by the weakened plate relative to a solid plate. At the end of the paper an example illustrating the application of this theory to a practical problem is discussed.

Theoretical Investigation

Limiting the discussion to simply supported rectangular plates, the general problem requires the determination of the deflection and rotation of a plate subjected to a local moment. In a general case, for the plate shown in Fig. 1, it is necessary to know the deflection and rotation of the plate caused by a local moment M_0 acting along a line such as AB. This solution has been previously reported.[3] However, in order to gain a necessary understanding of this solution it is briefly reviewed below.

The deflection w of a plate is governed by the well known differential equation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (1)$$

where q is the intensity of the load; and D, the flexural rigidity of the plate, is $Et^3/12(1-\mu^2)$. In the latter expression, E is the elastic modulus, t is the thickness of the plate and μ is Poisson's ratio.

The deflection w for a uniformly distributed load q acting over, for example, the shaded portion of the plate shown in the figure may be expressed in a series form.[4] One of these series applies in the region $y<\eta_1$; another in the region $y<\eta_2$; and a third series applies to the portion of the plate in the interval $\eta_1 < y < \eta_2$. By expanding the applied load q into a Fourier series and making use of the boundary and continuity conditions, all of the necessary constants for the above series may be determined.[5] This yields a solution for the deflection of a plate caused by a uniformly distributed load acting over the area $a_1 b_1$, shown shaded in Fig. 1. Then, by letting a_1 approach zero, as $a_1 b_1 q = P$ is maintained

3. "Effective Stiffness of a Plate Subjected to a Local Edge Moment," by I. A. Mohammed and E. P. Popov (publication pending). Proceedings, Second U. S. National Congress of Applied Mechanics.
4. "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill, 1940, pp. 147, 150-152.
5. See footnotes 3 and 4.

constant, one obtains an expression for w caused by a line load P acting over a distance b_1.

Upon introducing in an interval $\eta_1 < y < \eta_2$ two adjacent equal line loads, of magniture P each acting in opposite directions, the deflection of the plate may be obtained by superposition. Then, by letting the distance λ between the two line loads shrink to zero, as $P\lambda = M_o$ is maintained constant, one obtains an expression for w caused by a moment M_o uniformly distributed along a distance b_1.[6] The resulting expression is very involved. See Eqs. 8 and 9 in the Appendix. However, it greatly simplifies if the length b of the plate is made very large, i.e., $b \to \infty$, and the moment M_o is applied at $c = 0$, $\eta = 0$. For this special case it reduces to:

$$W = \frac{M_o a^2}{b_1 D \pi^3} \sum_{m=1}^{\infty} \frac{1}{m^3} \sin \frac{m\pi x}{a} \left\{ 2 + \left[\frac{m\pi y}{a} \sinh \frac{m\pi y}{a} - \left(2 + \frac{m\pi b_1}{2a}\right) \cosh \frac{m\pi y}{a} \right] e^{-\frac{m\pi b_1}{2a}} \right\} \qquad (2)$$

where m is an integer, and the meaning of the other terms is previously defined, or is indicated in Fig. 1.

For practical applications Eq. 2 is particularly significant. It gives the deflection w for a simply supported long plate loaded with a local moment M_o along the line JK in Fig. 1. This case closely corresponds to a bearing wall into which a horizontal beam of width b_1 is framed.

A study of Eq. 2, and analogous expressions given in the Appendix, shows that significant deflections of the plate occur in the close proximity of the loaded strip $y = \pm b_1/2$. At y-distances comparable to the width a of the plate, the deflections become small. This may be seen by examining Fig. 2 where deflection variation, due to concentrated moment M_o applied at $x = y = 0$, along several values of x = constant is shown. The computations were based on the set of equations for concentrated moment given in the Appendix, Eqs. 6a and 7a.

A further study of the partial derivative of Eq. 2 with respect to x shows that for a moment M_o, distributed along a distance b_1, a variation in the rotation of the edge of the plate within this distance takes place. For the purpose at hand, this rotation of the plate within the width b_1 at $x = 0$ is particularly significant. It is this rotation of the plate, a function of the applied moment M_o, that is directly related to the amount of restraint which a plate provides to a beam.

As stated above, for a uniformly distributed moment M_o the rotation of the plate at $x = 0$, derivable from Eq. 2, varies within the distance b_1. However, the moment is usually applied to the plate by means of a beam, of width b_1, having enormous rigidity and the condition of uniform distribution of the moment is not satisfied. Such a beam tends to enforce a uniform rotation of the plate within the width b_1. Therefore, there is a question as to which rotation of the plate should be used in establishing

6. See footnote 3.
7. See footnote 3.

an equivalent beam having the same stiffness for an applied moment as the plate. This may be resolved by investigating the rotation of the plate for the two extreme values of the plate's rotation within the distance b_1, i.e., at $y = 0$ and at $y = \pm b_1/2$. It was found that the difference between these two solutions is not very large and the correct answer, lying between these two extremes, may be reasonably well estimated. See Fig. 3.

The maximum slope of the loaded plate at $x = 0$ occurs at $y = 0$. It may be obtained by differentiating Eq. 2 with respect to x and then setting $x = y = 0$. On the other hand, the maximum slope of the elastic curve, for a simply supported beam loaded with an end moment M_o is given by the well known expression

$$\frac{dw}{dx} = \frac{M_o a}{3EI} \quad \frac{4M_o a}{EBd^3} \tag{3}$$

where a is the span, B is the beam's width, and d is its depth.

If the maximum slope or rotation of the plate is taken as a measure of the plate's stiffness to an applied moment, then the corresponding expression for the rotation of an equivalent simple beam must be equal to this maximum rotation. By formulating this equality, it is possible to relate the width B of the equivalent beam to the width b_1 as

$$B = K_o b_1 \tag{4}$$

where

$$\frac{1}{K_o} = \frac{3(1-\mu^2)}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[2 - \left(2 + \frac{m\pi b_1}{2a} \right) e^{-\frac{m\pi b_1}{2a}} \right] \tag{5}$$

Analogously, the rotation of the plate at $y = \pm b_1/2$ may be taken as a measure of the plate's stiffness. For this case the constant K_o in Eq. 4 must be replaced by $K_{b_1/2}$ and defined as

$$\frac{1}{K_{\frac{b_1}{2}}} = \frac{3(1-\mu^2)}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{2m^2} \left[2 - \left(2 + \frac{m\pi b_1}{a} \right) e^{-\frac{m\pi b_1}{a}} \right] \tag{5a}$$

The series defined by Eqs. 5 and 5a converge rather slowly. Direct evaluation of the constants K_o and $K_{b_1/2}$ is lengthy. To minimize the work in applications, the graph shown in Fig. 3 may be used. The value of K_o or $K_{b_1/2}$ may be easily obtained from this graph for a considerable range of values of the parameter b_1/a. However, as stated earlier, a beam framing into a wall has enormous rigidity, and it tends to enforce a uniform rotation of the edge of the wall within the distance b_1. Thus, neither K_o nor $K_{b_1/2}$ are the exact constants for the solution of the actual problem. However, for a practical problem, with a sufficient degree of accuracy, the true value of the constant may be taken as $K_{av} = \frac{(K_o + K_{b_1/2})}{2}$. This follows from the assumption of a linear variation of

the rotation of the edge of the plate from $y = 0$ to $y = \pm b_1/2$. This approximation becomes somewhat inaccurate for very large values of b_1. Thus, for applications, one should compute the value of the width B from the relation

$$B = K_{av} b_1 \tag{4a}$$

The curve for K_{av} is shown in Fig. 3.

Experimental Investigation

Since door and window openings almost invariably occur in bearing walls, the amount of restraint given to beams by such walls is an important problem related to the one discussed above. Mathematical difficulties precluded the analytical solution of this problem, so a limited aspect was studied experimentally. The performed experiments are described below. The analysis of the experimental data permitted the construction of curves which enable one to determine the restraint provided to a beam by a wall weakened by various openings as a fraction of the restraint given by a similar solid wall. In all cases the horizontal width of the opening investigated was equal to the height of the wall.

A general view of the test set up is shown in Fig. 4. A wall was simulated with a 24" x 1/4" x 8'-0" steel plate, which was hung vertically by four cables. At top and bottom, "simple supports" for the plate were provided by 1/2" x 1/16" x 13" aluminum strips spaced approximately 1-1/2" on centers, and attached to the plate with machine screws. Since the plate was long in relation to its width, the two short edges of the plate were left free. The moment at the middle of the top edge could be applied by means of six levers, close-up view in Fig. 5. Through a pulley arrangement, by applying known upward forces on one end of the levers and downward known forces on the other end, it was possible to apply external edge moments of desired magnitudes. The magnitudes of the known forces were controlled by placing weights on pans, as may be seen in the rear of Fig. 4. The levers were so constructed that the moment was applied within 7/16" of the top of the plate. The rotation of the plate caused by the applied moments was measured by very sensitive clinometers which were mounted on small rods, as may be seen in Fig. 5.

In several tests, in order to study the effect of various widths of beams framing into the plate, the outer two or the outer four levers were left unloaded. In some tests the loaded levers were rigidly clamped together to enforce a uniform rotation of the edge of the plate along the loaded edge. These cases more nearly correspond to the actual beams framing into a wall.

Systematic experiments were made using the above described apparatus. The initial tests were performed on a solid plate. Then a saw cut 24 in. long was made 10 in. above and parallel to the bottom of the plate. Subsequently analogous tests were made by gradually increasing the height of the opening in 3 in. steps. In this manner data for 3, 6, 9, 12, 15, and 18 in. high rectangular openings were obtained. In all cases

the width of the opening was maintained at 24 in. The location of these openings is shown diagrammatically in Fig. 6.

In each trial the rotation of the plate at the middle of the top edge and at $y = 3\text{-}1/16$ in. was measured. At any one time, two, four, or six levers were loaded. The clinometer readings were taken in the unloaded condition, at half-load, full load, and again after the loads were removed. In addition to the readings taken for the rotation of the plate's edge, the deflections of the plate at several locations were measured with a special floating meter, which may be seen in Fig. 7. In this particular photograph the meter is located at a distance $a = 24$ in. from the center of the plate. (The opening in the plate which may be seen in Fig. 4 is the largest one investigated.)

The deflections of the solid plate used in these experiments caused by an effective moment $M_o = 930$ in.-lb. applied by means of six levers is shown in Fig. 8. Three curves, giving the deflection at $y = a = 24$ in., $y = a/2 = 12$ in., and $y = 0$, show the results when all six levers acted independently; one curve, for $y = 0$, is shown when all levers are rigidly clamped together. It is significant to note that even in this case of a relatively widely loaded region, the difference in deflection at $y = 0$ for the levers acting independently or clamped together is rather small.

The variation in the rotation of the plate's edge at the middle and near the edge of the loaded region may be seen from Fig. 9. The curves show the effect of the six levers acting independently with a total effective moment of 930 in.-lb. The readings were made using the two clinometers shown in Fig. 5. The outer clinometer was mounted at $y =$ 3-1/16 in., whereas the edge of the exterior lever was at $y = 2\text{-}13/16$ in. Therefore, the difference between the two curves exaggerates the actual difference in the rotation at the middle and at the edge of the exterior lever.

The rotation at the middle of the top plate for progressively larger rectangular openings is plotted in Fig. 10. From this diagram the difference in the maximum rotation may be seen for the six levers acting independently and clamped together. In both cases the effective applied moment was 930 in.-lb. For narrower beams, i.e., for four or two lever loading, the difference between the corresponding curves is smaller.

The various observed maximum rotations of the plate were compared with the maximum rotation caused by the same loading condition for a solid plate. This permitted the calculation of the restraint provided by the plate with different size openings as a percentage of the restraint given by a solid plate. The height h of the opening was expressed as a fraction of the height a of the wall. Using these dimensionless parameters, the curves shown in Fig. 11 were established. Each curve corresponds to the particular ratio of the width b_1 of a beam framing into a wall of height a. By interpolating between these curves a considerable range of possible beam widths and opening sizes may be analyzed. The curve shown in Fig. 11 for the six levers acting independently is of some theoretical interest.

Comparison of Experimental and Theoretical Results

The results of the experiments with a solid plate loaded with levers acting independently may be compared with the established analytical expressions. For this purpose the mechanical properties of the steel plate were determined, and the sources of the experimental errors were analyzed and compensated for in the comparison.

A sample of the steel plate 1.00 in. by 0.251 in. by 24 in. was tested in pure bending, and, from the data, the modulus of the elasticity E was found to be 31.18 x 10^6 psi. Poisson's ratio μ was assumed to be 0.30. These values of E and μ were used in all calculations.

It was felt that the readings obtained with the clinometers and dial gages were reliable. However, the cables used in the loading arrangement were not sufficiently flexible, and there was a considerable amount of friction in the pulleys. Therefore, it was necessary to calibrate the loading system and to determine accurately the magnitude of the actual forces delivered to the ends of the levers. This calibration showed that approximately 26.3% of the force originating from the loads in the pans was lost in pulley friction and bending of the cables. All other sources of experimental error were rather small. Nevertheless an attempt to compensate for them was also made.

The aluminum strips at the top and bottom of the plate actually provided a little restraint, i.e., the support was not truly "simple". The amount of restraint given to the plates was found to be about 1.0% which indicates that this detail of the apparatus was very good.

Since the plate was actually hung vertically from four cables, the weight of the plate tended to straighten it out and the observed deflections and rotations were reduced. It is estimated that this effect also amounted to approximately 1.0%. The fact that the moment could not be applied precisely at the edge of the plate also contributed to the reduction of the applied moment to the extent of 0.2%. It was felt that the other sources of experimental error could be ignored.

Based on the above analysis of the possible experimental errors, it was concluded that the effective moment applied to the plate was reduced by approximately 29% in comparison with the moment which presumably originated from the loads in the pans. Therefore, if each of the twelve pans were loaded with a weight of 10 lbs. and the moment arm for each force in the lever arrangement were 10 in., the moment applied to the top of the plate was not 12 x 10 x 10 = 1,200 in.-lb. but 1,200/1.29 = 930 in.-lb. This is the magnitude previously cited. Conversely, if the calculations for the deflection or rotation of the plate are based on an applied moment of 200 in.-lb. per lever, the observed values must be multiplied by a factor of 1.29.

Since the analytical expressions derived are based on the assumption of a uniformly distributed moment, the comparison must be made with the experiments where the levers act independently (not clamped together). The nature of the agreement found between the calculated and the adjusted observed values may be seen from Table 1. The calculations are based on the assumption that each fully loaded lever applies a moment of 200 in.-lb. to the top edge of the plate.

Table 1

Deflection of the Plate

Deflections are given in 0.001 of an inch. All levers act independently.
Columns marked "A" give the calculated values of deflection.
Columns marked "B" give the observed values of deflection × 1.29.

$\frac{x}{a}$	6 levers loaded						Inner four levers loaded						Inner 2 levers loaded					
	y = 0		y = a/2		y = a		y = 0		y = a/2		y = a		y = 0		y = a/2		y = a	
	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1/8	23.3	23.2	8.0	7.7	2.3	2.2	16.1	16.0	5.2	4.9	1.5	1.5	8.3	8.9	2.6	2.5	.8	.9
1/4	30.9	31.6	13.9	13.9	4.2	4.0	21.4	21.3	9.2	8.8	2.8	2.6	10.6	11.4	4.6	4.0	1.4	1.3
3/8	31.9	32.2	17.0	16.6	5.5	5.3	21.5	21.4	11.3	10.6	3.6	3.6	10.8	11.5	5.6	5.2	1.8	1.7
1/2	29.3	29.0	17.2	16.6	5.8	6.2	19.7	19.5	11.4	10.7	3.8	3.9	9.9	10.3	5.7	5.4	1.9	1.9
5/8	23.9	--	14.9	--	5.3	--	16.0	--	9.9	--	3.5	--	8.0	--	5.0	--	1.8	--
3/4	16.7	16.0	10.9	10.5	4.0	3.9	11.2	10.5	7.3	7.1	2.6	2.3	5.6	5.4	3.6	3.9	1.3	1.0
7/8	8.7	--	5.8	--	2.1	--	5.8	--	3.8	--	1.4	--	3.0	--	1.9	--	.7	--
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

An examination of the data given in Table 1 shows excellent agreement between the calculated and the adjusted observed values of the plate's deflection. Also, equally good agreement between the calculated and experimental values was found for the maximum rotation of the plate, as may be seen from Table 2.

Table 2

Rotation of the Plate at $x = 0$, $y = 0$

Rotation is given in radians. All levers act independently.

6 levers loaded		Inner 4 levers loaded		Inner 2 levers loaded	
Calculated	Observed x 1.29	Calculated	Observed x 1.29	Calculated	Observed x 1.29
0.0125	0.0126	0.00946	0.00967	0.00570	0.00603

Illustrative Example

Compare the restraint given by the vertical member to identical horizontal beam for the two cases shown in Fig. 12. In the first instance a 12 in. wide beam frames into a 12" x 10" column, simply supported at the bottom. In the second case, the beam frames into a wall with an opening. The wall is simply supported along all edges. The column and the wall are made of concrete, for which $\mu = 1/6$.

In the second case the ratio of the beam's width to the height of the wall is $b_1/a = 1/10 = 0.1$. For this ratio, from Fig. 3, the constant K_{av} is 7.10. However since $\mu = 1/6$, this constant must be multiplied by the factor $0.91/[1 - (1/6)^2] = 0.936$; hence $K_{av} = 7.10 \times 0.936 = 6.64$. If this horizontal beam were framed into a solid wall, an equivalent vertical beam (column) providing the same amount of restraint as the solid wall, according to Eq. 4a, would have the width equal to

$$B = K_{av} b_1 = 6.64 \times 1 = 6.64 \text{ ft.}$$

However, since the wall is weakened by the opening, the restraint is reduced to the percentage which may be determined from Fig. 11. Thus, using the ratio of the opening's height h to the height of the wall a, i.e., $h/a = 3.75/10 = 0.375$, one obtains from Fig. 11 the restraint factor of 73%. Therefore, the width of the equivalent vertical beam sought in this case is $6.64 \times 0.73 = 4.85$ ft. Hence, a wall with an opening such as the one shown in Fig. 12b provides 4.85 times as much restraint to the horizontal beam as does the wall-column shown in Fig. 12a.

CONCLUDING REMARKS

The stiffness of a simply supported solid plate subjected to a local edge moment was discussed in this paper. A graph, Fig. 3, was established for determining the width of a simply supported beam which has the same effective stiffness as the plate. However, in practice, more

than one beam frames into the wall at various spacings. In such circumstances, when the edge moments are simultaneously applied to the wall by several beams, the rotation of the wall, at the location of any one beam, is influenced by the other beams. This may be accounted for by using the influence lines shown in Fig. 13. The curves shown in this figure are the influence lines for the rotation of the top edge of a simply supported long plate subjected to a moment, uniformly distributed along a distance b_1 located at the middle of this edge. These curves are for three different ratios of b_1/a; for other values of b_1/a interpolation and extrapolation may be used. To establish these curves $\partial w/\partial x$ at $x=0$ for several values of y were calculated. The expression for w is given by Eqs. 8a,8b, and 9a given in the Appendix. The series for $\partial w/\partial x$ converge slowly and a large number of terms was used.

For a particular ratio of b_1/a the corresponding influence line shown in Fig. 13 may be used as follows: First, the positions of the adjoining beams are marked off along the abscissa. Then, the ordinates to the curve are scaled off at each one of the points corresponding to the location of a beam. The sum of all of the ordinates so found is the magnification factor by which the maximum rotation of the plate at the selected beam is increased due to the presence of the adjoining beams. By dividing the width B given by Eq. 4a by this magnification factor, the width of a simply supported beam having a stiffness equivalent to a plate loaded with a number of discrete moments is obtained.

The curves of Fig. 11, based on experiments, enable one to determine the percentage of the restraint provided by a plate with an opening as compared with the restraint of a solid plate. The width of opening for which these curves apply must be equal to the height of the wall. Additional experiments need to be performed to enable one to analyze walls with different widths of openings.

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APPENDIX

For reference purposes several formulas are given below.

A. Concentrated Moment

The following formulas apply to a simply supported plate subjected to a concentrated moment M_o, see Fig. 14a.

The deflection w along $y=\eta$ is:

$$w = \frac{M_o a}{2D\pi^2} \sum_{m=1}^{\infty} \frac{\cos\frac{m\pi c}{a} \sin\frac{m\pi x}{a}}{m^2} \Big[\cosh^2\frac{m\pi\eta}{a}\Big(\tanh\alpha_m - \frac{\alpha_m}{\cosh^2\alpha_m}\Big)$$

$$-\sinh^2\frac{m\pi\eta}{a}\Big(\coth\alpha_m + \frac{\alpha_m}{\sinh^2\alpha_m}\Big) - 2\frac{m\pi\eta}{a}\sinh\frac{m\pi\eta}{a}\cosh\frac{m\pi\eta}{a}\Big(\tanh\alpha_m - \coth\alpha_m\Big)\Big] \quad (6)$$

where $\alpha_m = \frac{m\pi b}{2a}$

when $c = 0$, $\eta = 0$, and $b \to \infty$, Eq. 6 reduces to:

$$w = \frac{M_o a}{2D\pi^2} \sum_{m=1}^{\infty} \frac{\sin\frac{m\pi x}{a}}{m^2} \quad (6a)$$

The deflection w' for the region $y>\eta$, or the deflection w'' for the region $y<\eta$, is:

$$\left.\begin{matrix} w' \\ w'' \end{matrix}\right> = \frac{M_0 a}{2D\pi^2}\sum_{m=1}^{\infty}\frac{\cos\frac{m\pi c}{a}\sin\frac{m\pi x}{a}}{m^2}$$

$$\left\{+\left[\left(\tanh\alpha_m\cosh\frac{m\pi\eta}{a}\pm\sinh\frac{m\pi\eta}{a}\right)\mp\frac{m\pi\eta}{a}\left(\cosh\frac{m\pi\eta}{a}\pm\tanh\alpha_m\sinh\frac{m\pi\eta}{a}\right)-\alpha_m\frac{\cosh\frac{m\pi\eta}{a}}{\cosh^2\alpha_m}\right]\cosh\frac{m\pi y}{a}\right.$$

$$+\left[\mp\left(\cosh\frac{m\pi\eta}{a}\pm\coth\alpha_m\sinh\frac{m\pi\eta}{a}\right)+\frac{m\pi\eta}{a}\left(\coth\alpha_m\cosh\frac{m\pi\eta}{a}\pm\sinh\frac{m\pi\eta}{a}\right)-\alpha_m\frac{\sinh\frac{m\pi\eta}{a}}{\sinh^2\alpha_m}\right]\sinh\frac{m\pi y}{a}$$

$$\left.-\left(\tanh\alpha_m\cosh\frac{m\pi\eta}{a}\pm\sinh\frac{m\pi\eta}{a}\right)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\pm\left(\cosh\frac{m\pi\eta}{a}\pm\coth\alpha_m\sinh\frac{m\pi\eta}{a}\right)\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right\} \quad (7)$$

when $c=0$, $\eta=0$ and $b\to\infty$, equation 7 reduces to:

$$\left.\begin{matrix} w' \\ w'' \end{matrix}\right> = \frac{M_0 a}{2D\pi^2}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x}{a}}{m^2}\left(1\pm\frac{m\pi y}{a}\right)e^{\pm\frac{m\pi y}{a}}$$

$$(7a)$$

$$w = \frac{2M_0a^2}{b_1D\pi^3}\sum_{m=1}^{\infty}\frac{\cos\frac{m\pi c}{a}\sin\frac{m\pi x}{a}}{m^3}\left\{+1\right.$$

B. Uniformly Distributed Moment

The following formulas apply for a simply supported plate subjected to a moment M_0 uniformly distributed along a distance b_1, see Fig. 14b.

The deflection w for $\eta_1 < y < {}_2$ is:

$$-\left[\left(\cosh\frac{m\pi b_1}{2a} - \tanh\alpha_m \sinh\frac{m\pi b_1}{2a}\right) + \frac{m\pi b_1}{4a}\left(\tanh\alpha_m \cosh\frac{m\pi b_1}{2a} - \sinh\frac{m\pi b_1}{2a}\right) + \alpha_m \frac{\sinh\frac{m\pi b_1}{2a}}{2\cosh^2\alpha_m}\right]\cosh\frac{m\pi\eta}{a}\cosh\frac{m\pi y}{a}$$

$$+\left[\left(\cosh\frac{m\pi b_1}{2a} - \coth\alpha_m \sinh\frac{m\pi b_1}{2a}\right) + \frac{m\pi b_1}{4a}\left(\coth\alpha_m \cosh\frac{m\pi b_1}{2a} - \sinh\frac{m\pi b_1}{2a}\right) - \alpha_m \frac{\sinh\frac{m\pi b_1}{2a}}{2\sinh^2\alpha_m}\right]\sinh\frac{m\pi\eta}{a}\sinh\frac{m\pi y}{a}$$

$$+\left[\left(\cosh\frac{m\pi b_1}{2a} - \tanh\alpha_m \sinh\frac{m\pi b_1}{2a}\right)\frac{m\pi\eta}{2a} - \left(\cosh\frac{m\pi b_1}{2a} - \coth\alpha_m \sinh\frac{m\pi b_1}{2a}\right)\frac{m\pi y}{2a}\right]\sinh\frac{m\pi\eta}{a}\cosh\frac{m\pi y}{a}$$

$$-\left[\left(\cosh\frac{m\pi b_1}{2a} - \coth\alpha_m \sinh\frac{m\pi b_1}{2a}\right)\frac{m\pi\eta}{2a} - \left(\cosh\frac{m\pi b_1}{2a} - \tanh\alpha_m \sinh\frac{m\pi b_1}{2a}\right)\frac{m\pi y}{2a}\right]\cosh\frac{m\pi\eta}{a}\sinh\frac{m\pi y}{a}\Bigg\} \qquad (8)$$

when c = 0, $\eta = 0$ and $b \to \infty$, Eq. 8 reduces to:

$$w = \frac{M_0 a^2}{b_1 D \pi^3}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x}{a}}{m^3}\left\{2 + \left[\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} - \left(2 + \frac{m\pi b_1}{2a}\right)\cosh\frac{m\pi y}{a}\right]e^{-\frac{m\pi b_1}{2a}}\right\} \qquad (8a)$$

when y = 0, Eq. 8a reduces to:

$$w = \frac{M_0 a^2}{b_1 D \pi^3} \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi x}{a}}{m^3} \left[2 - \left(2 + \frac{m\pi b_1}{2a} \right) e^{-\frac{m\pi b_1}{2a}} \right] \tag{8b}$$

The deflection w' for the region $y \geq \eta_2$, or the deflection w'' for the region $y \leq \eta_1$, is:

$$\left.\begin{matrix} w' \\ w'' \end{matrix}\right\rangle = \frac{M_0 a^2}{b_1 D \pi^3} \sum_{m=1}^{\infty} \frac{\cos \frac{m\pi c}{a} \sin \frac{m\pi x}{a} \sinh \frac{m\pi b_1}{2a}}{m^3}$$

$$\left\{ + \left[\left(2 - \frac{m\pi b_1}{2a} \coth \frac{m\pi b_1}{2a} \right) \left(\tanh \alpha_m \cosh \frac{m\pi\eta}{a} \pm \sinh \frac{m\pi\eta}{a} \right) \mp \frac{m\pi\eta}{a} \left(\cosh \frac{m\pi\eta}{a} \pm \tanh \alpha_m \sinh \frac{m\pi\eta}{a} \right) - \alpha_m \frac{\cosh \frac{m\pi\eta}{a}}{\cosh^2 \alpha_m} \right] \cosh \frac{m\pi y}{a} \right.$$

$$+ \left[\left(\mp 2 \pm \frac{m\pi b_1}{2a} \coth \frac{m\pi b_1}{2a} \right) \left(\cosh \frac{m\pi\eta}{a} \pm \coth \alpha_m \sinh \frac{m\pi\eta}{a} \right) + \frac{m\pi\eta}{a} \left(\coth \alpha_m \cosh \frac{m\pi\eta}{a} \pm \sinh \frac{m\pi\eta}{a} \right) - \alpha_m \frac{\sinh \frac{m\pi\eta}{a}}{\sinh^2 \alpha_m} \right] \sinh \frac{m\pi y}{a}$$

$$\left. - \frac{m\pi y}{a} \left(\tanh \alpha_m \cosh \frac{m\pi\eta}{a} \pm \sinh \frac{m\pi\eta}{a} \right) \sinh \frac{m\pi y}{a} \pm \frac{m\pi y}{a} \left(\cosh \frac{m\pi\eta}{a} \pm \coth \alpha_m \sinh \frac{m\pi\eta}{a} \right) \cosh \frac{m\pi y}{a} \right\} \tag{9}$$

when c = 0, $\eta = 0$ and $b \to \infty$, Eq. 9 reduces to:

$$\left.\begin{matrix} w' \\ w'' \end{matrix}\right\rangle = \frac{M_0 a^2}{b_1 D \pi^3} \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sinh \frac{m\pi b_1}{2a}}{m^3} \left(2 \pm \frac{m\pi y}{a} - \frac{m\pi b_1}{2a} \coth \frac{m\pi b_1}{2a} \right) e^{\mp \frac{m\pi y}{a}} \tag{9a}$$

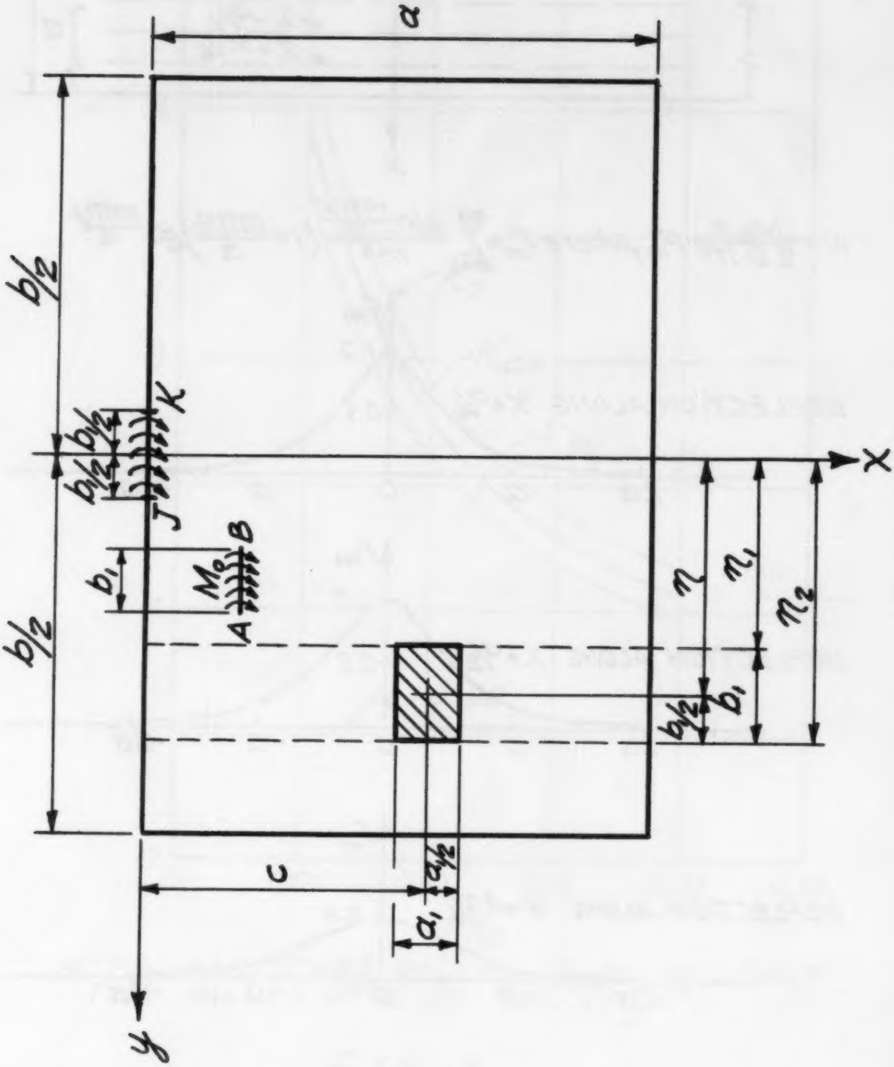
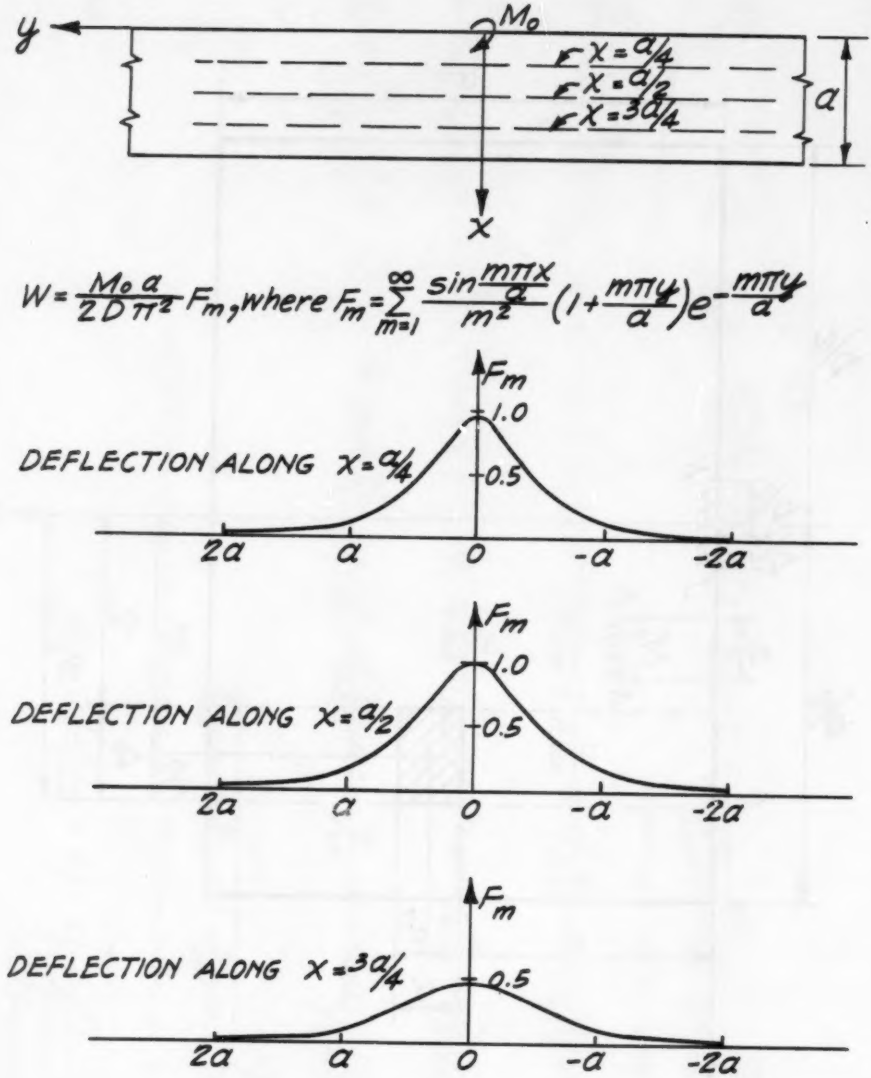
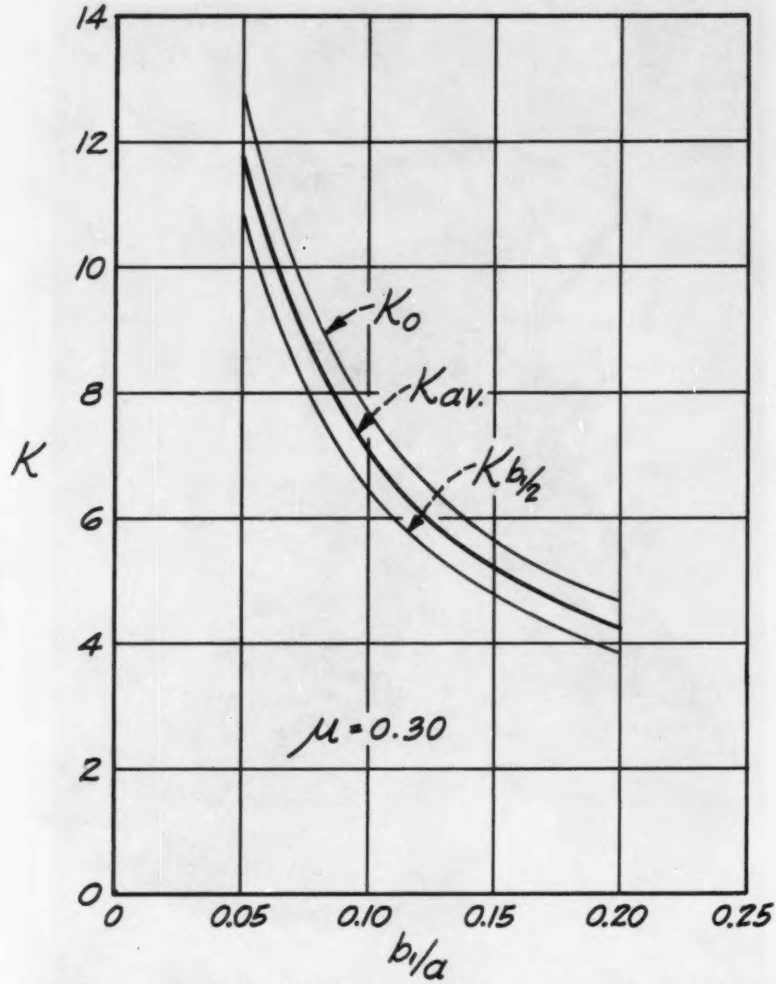


FIG. 1



y
M_0
$x = a/4$
$x = a/2$
$x = 3a/4$
a
x
$W = \frac{M_0 a}{2D\pi^2} F_m$, where $F_m = \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi x}{a}}{m^2} \left(1 + \frac{m\pi y}{a}\right) e^{-\frac{m\pi y}{a}}$
DEFLECTION ALONG $x = a/4$
F_m
1.0
0.5
2a
a
0
-a
-2a
DEFLECTION ALONG $x = a/2$
F_m
1.0
0.5
2a
a
0
-a
-2a
DEFLECTION ALONG $x = 3a/4$
F_m
0.5
2a
a
0
-a
-2a

FIG. 2



NOTE: FOR OTHER VALUES OF μ, MULTIPLY THE VALUES OF K BY $0.91/(1-\mu^2)$.

FIG. 3

FIG. 4

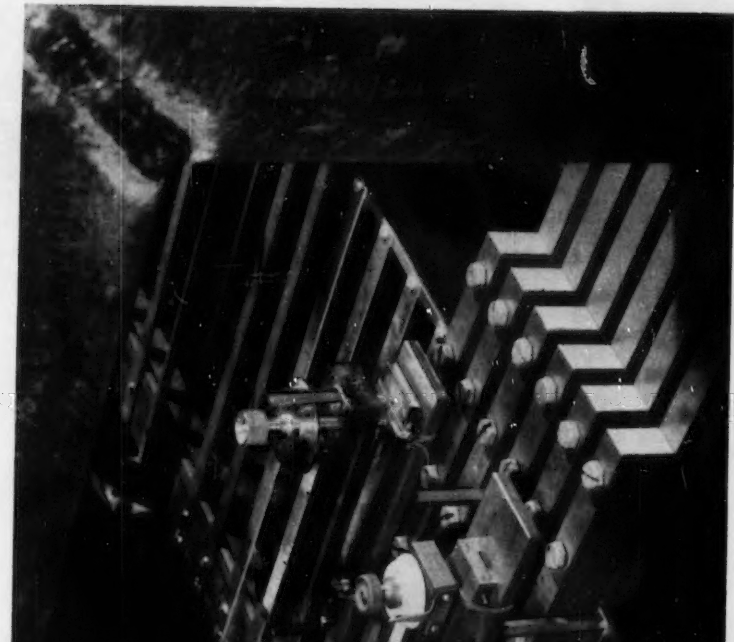


FIG. 5

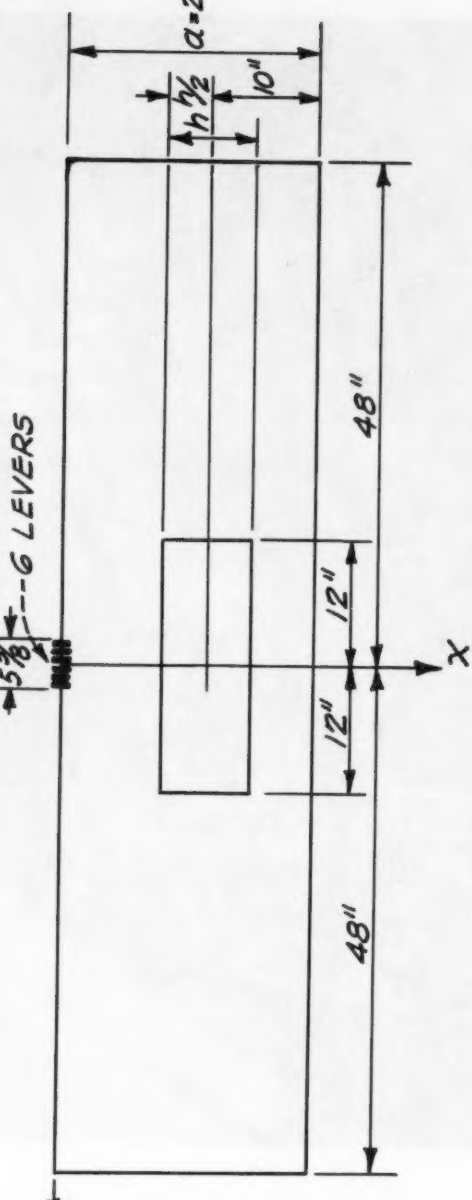


FIG. 6

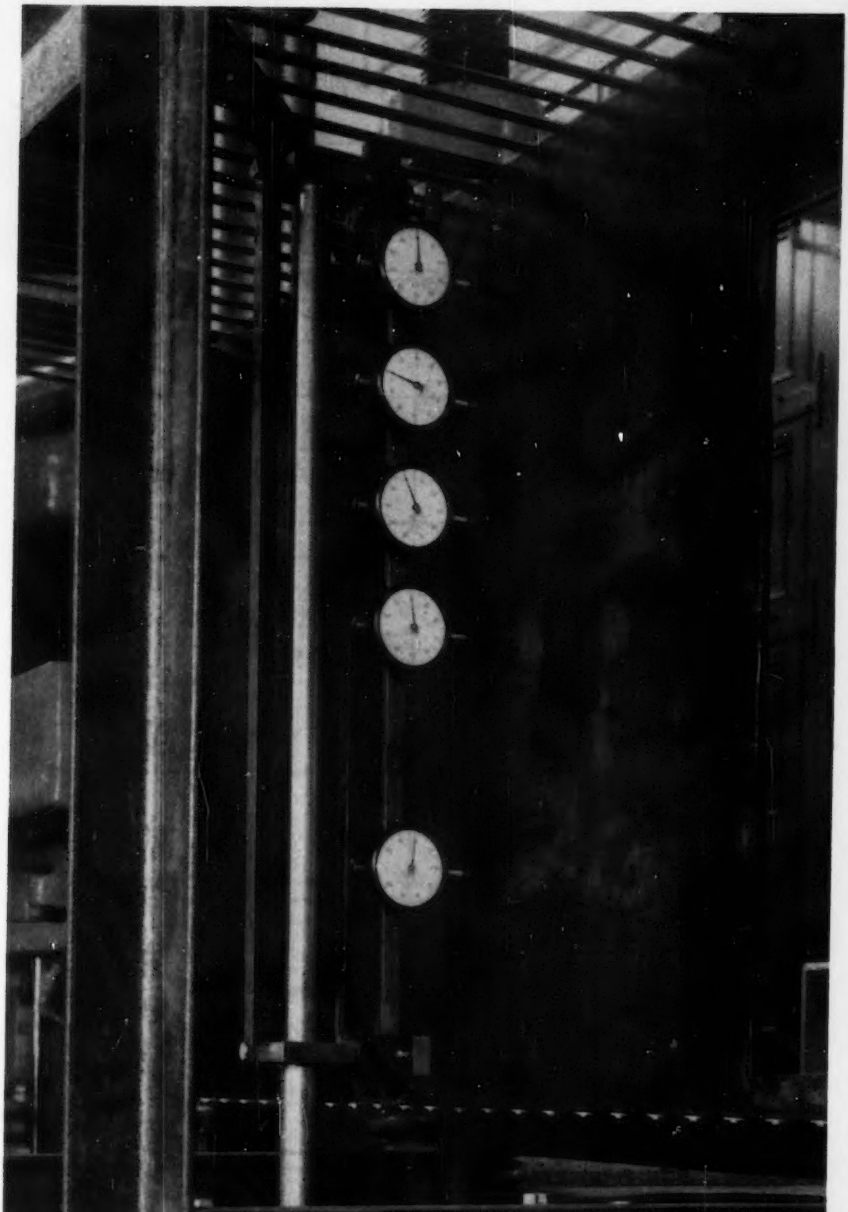
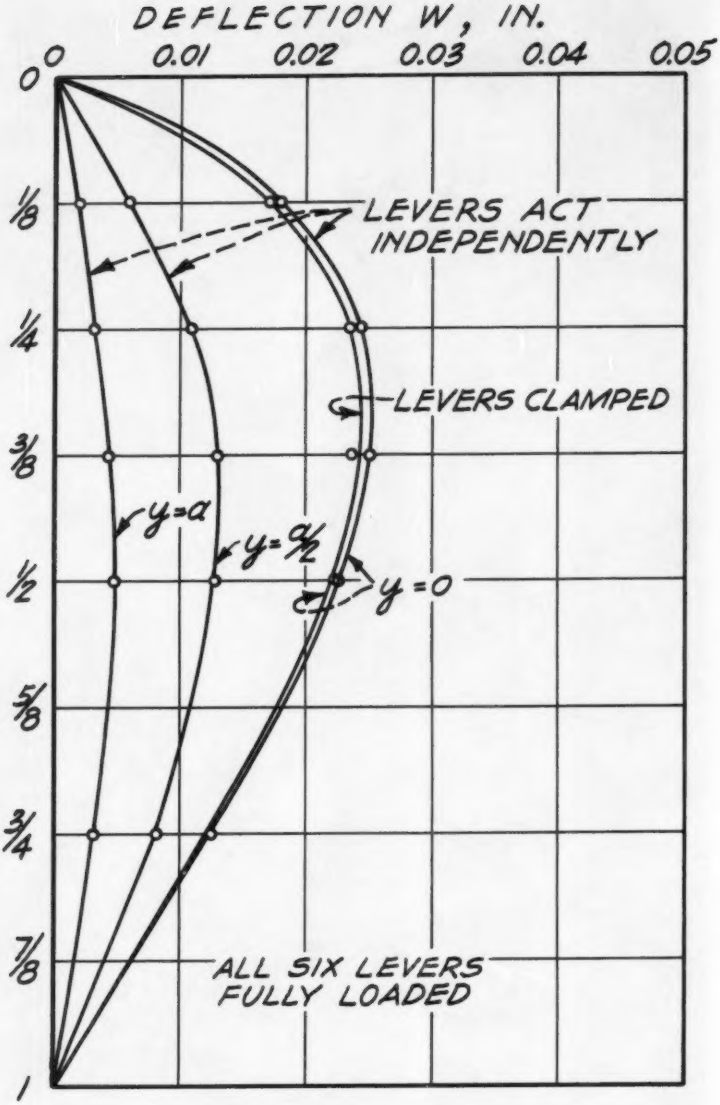


FIG. 7



DEFLECTION W, IN.
0
0.01
0.02
0.03
0.04
0.05
0
1/8
1/4
3/8
x/a
1/2
5/8
3/4
7/8
1
LEVERS ACT INDEPENDENTLY
LEVERS CLAMPED
y=a
y=a/2
y=0
ALL SIX LEVERS FULLY LOADED

FIG. 8

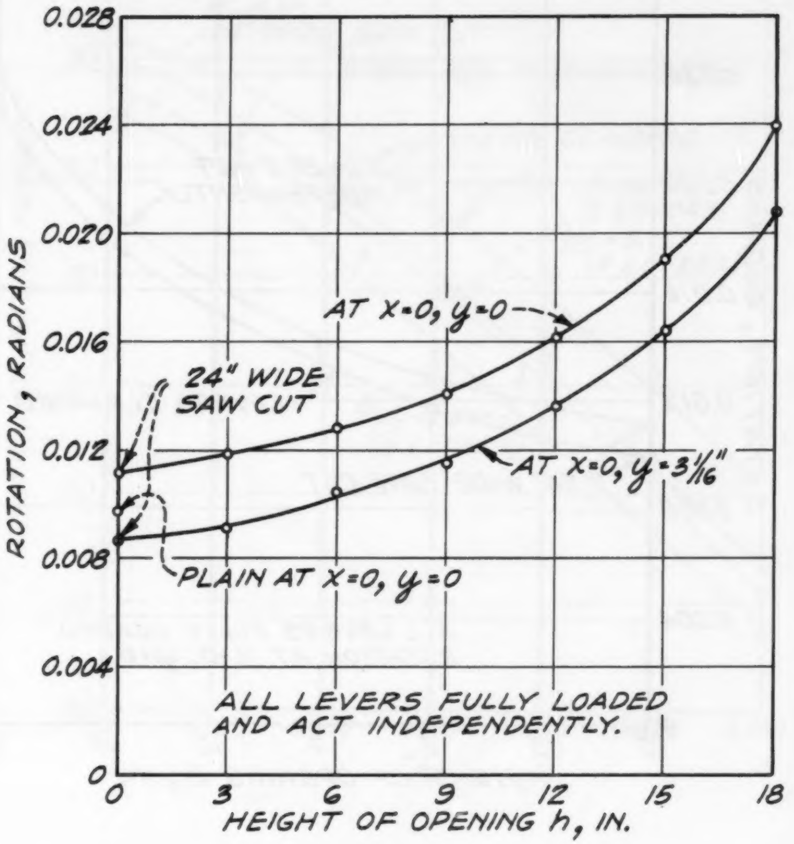

0.028
0.024
0.020
0.016
0.012
0.008
0.004
0
ROTATION, RADIANS
0
3
6
9
12
15
18
HEIGHT OF OPENING h, IN.
AT X=0, y=0
24" WIDE SAW CUT
AT X=0, y=3 1/16"
PLAIN AT X=0, y=0
ALL LEVERS FULLY LOADED AND ACT INDEPENDENTLY.

FIG. 9

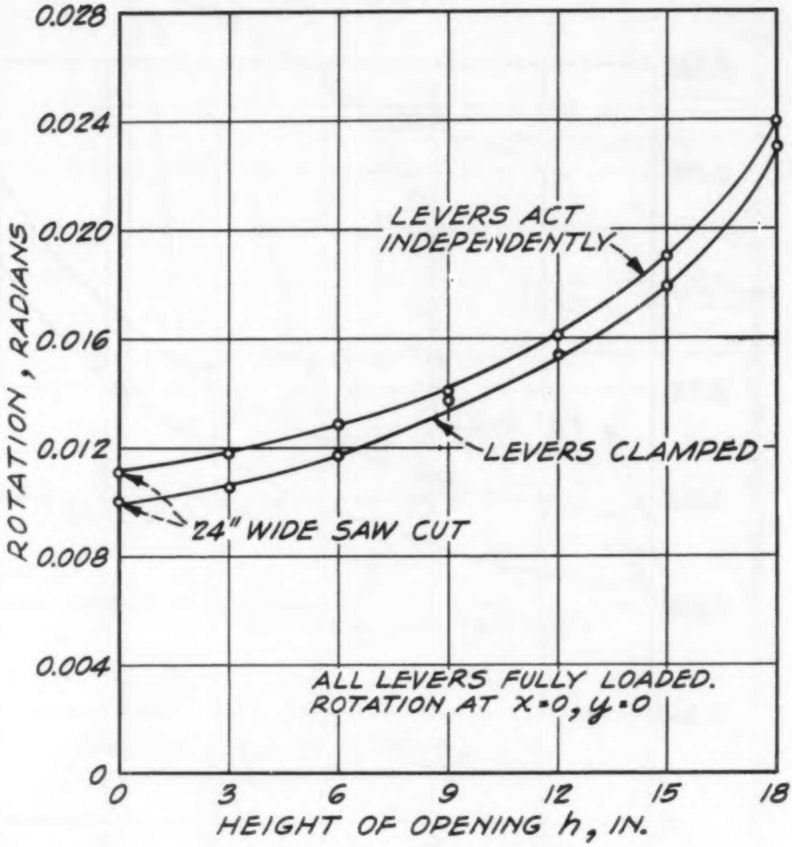
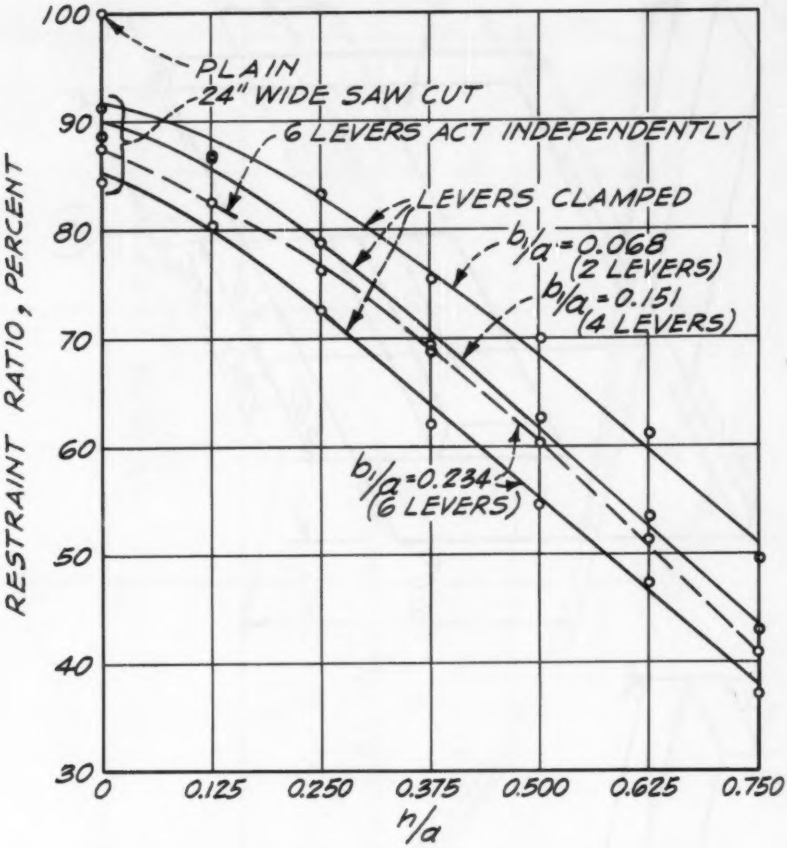

ROTATION, RADIANS
0.028
0.024
0.020
0.016
0.012
0.008
0.004
0
LEVERS ACT INDEPENDENTLY
LEVERS CLAMPED
24" WIDE SAW CUT
ALL LEVERS FULLY LOADED.
ROTATION AT x=0, y=0
0
3
6
9
12
15
18
HEIGHT OF OPENING h, IN.

FIG. 10



100
90
80
70
60
50
40
30
RESTRAINT RATIO, PERCENT
0
0.125
0.250
0.375
0.500
0.625
0.750
h/a
PLAIN
24" WIDE SAW CUT
6 LEVERS ACT INDEPENDENTLY
LEVERS CLAMPED
b1/a = 0.068
(2 LEVERS)
b1/a = 0.151
(4 LEVERS)
b1/a = 0.234
(6 LEVERS)

FIG. 11

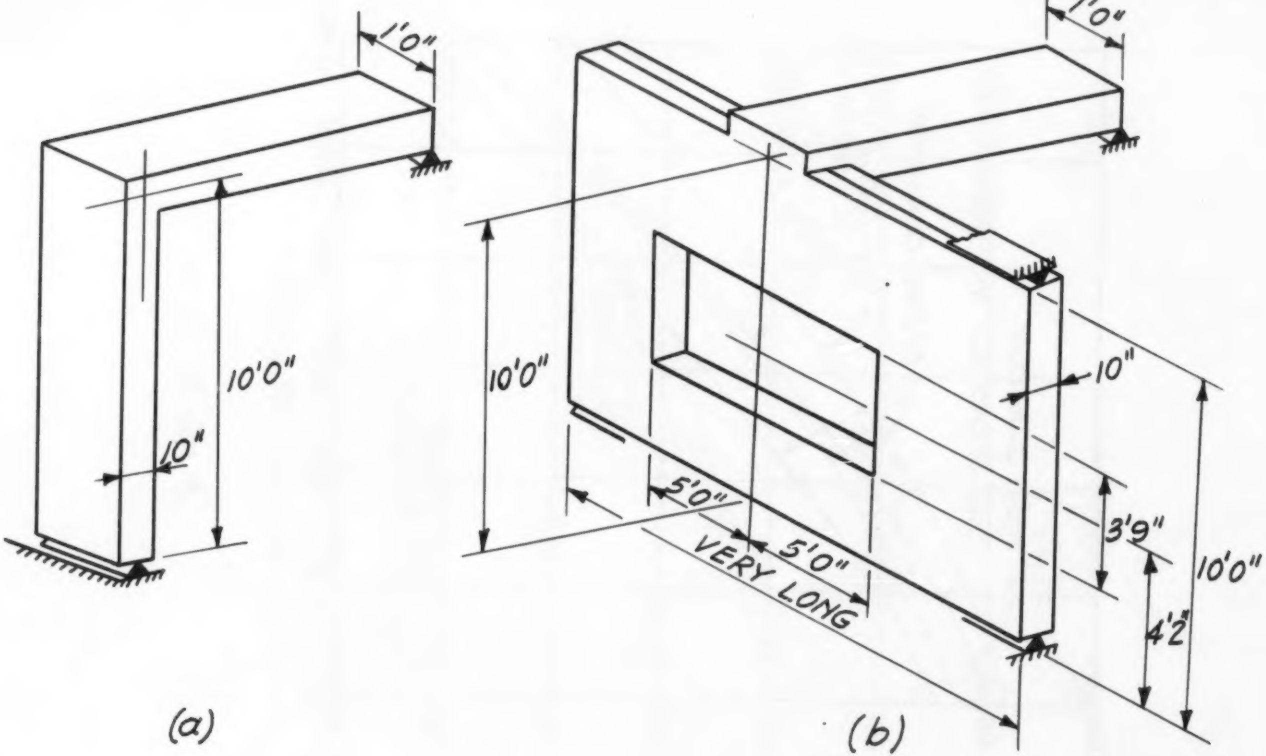


FIG. 12

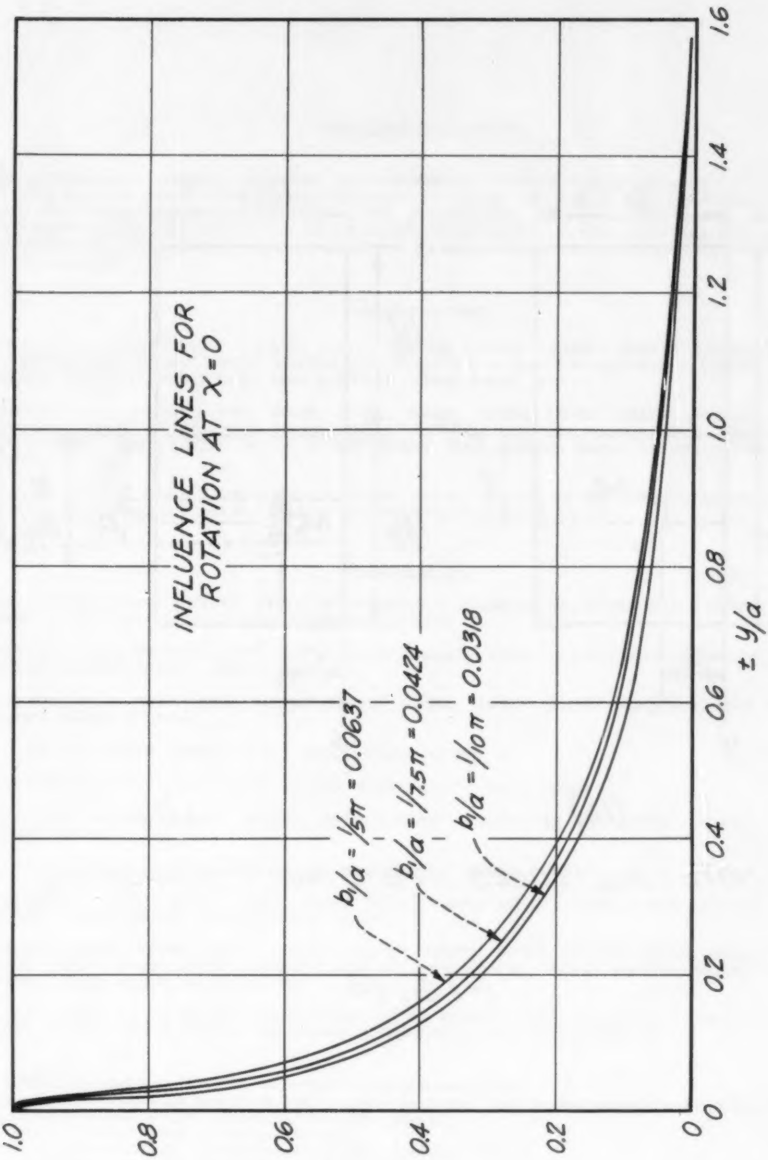
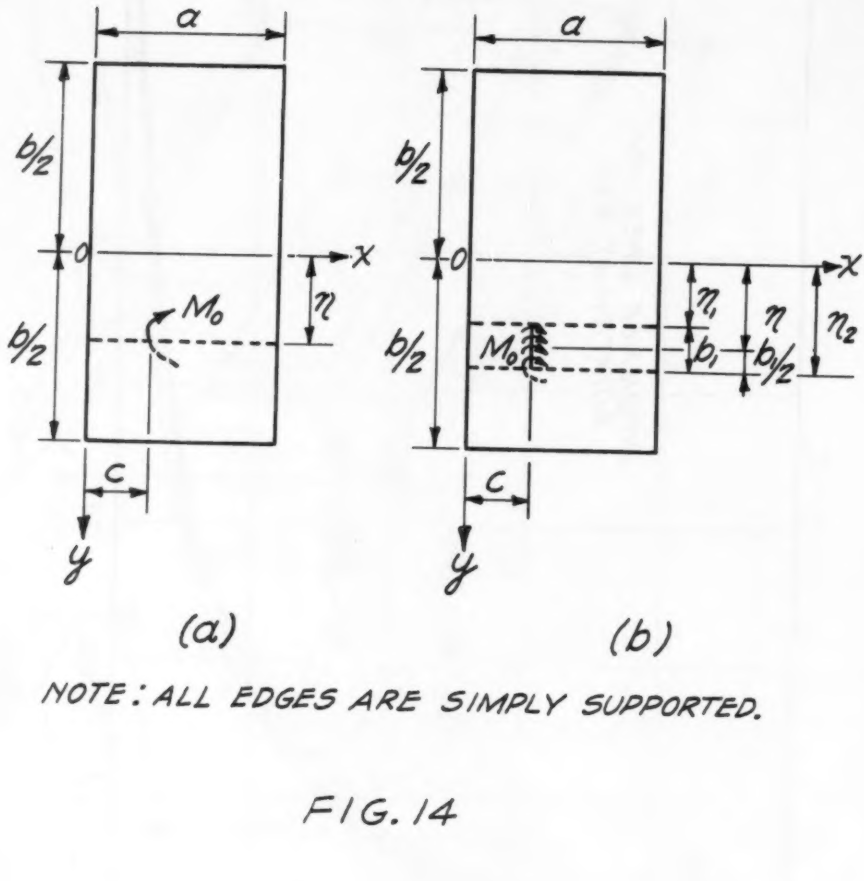

INFLUENCE LINES FOR
ROTATION AT X = 0
b/a = 1/5π = 0.0637
b/a = 1/7.5π = 0.0424
b/a = 1/10π = 0.0318
± y/a
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
0
0.2
0.4
0.6
0.8
1.0

FIG. 13



NOTE: ALL EDGES ARE SIMPLY SUPPORTED.

FIG. 14

PROCEEDINGS-SEPARATES

The technical papers published in the past year are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Separate Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

VOLUME 79 (1953)

OCTOBER:[b] 290(all Divs), 291(ST)[a], 292(EM)[a], 293(ST)[a], 294(PO)[a], 295(HY)[a], 296(EM)[a], 297(HY)[a], 298(ST)[a], 299(EM)[a], 300(EM)[a], 301(SA)[a], 302(SA)[a], 303(SA)[a], 304(CO)[a], 305(SU)[a], 306(ST)[a], 307(SA)[a], 308(PO)[a], 309(SA)[a], 310(SA)[a], 311(SM)[a], 312(SA)[a], 313(ST)[a], 314(SA)[a], 315(SM)[a], 316(AT), 317(AT), 318(WW), 319(IR), 320(HW).

NOVEMBER: 321(ST), 322(ST), 323(SM), 324(SM), 325(SM), 326(SM), 327(SM), 328(SM), 329(HW), 330(EM)[a], 331(EM)[a], 332(EM)[a], 333(EM)[c], 334(EM), 335(SA), 336(SA), 337(SA), 338(SA), 339(SA), 340(SA), 341(SA), 342(CO), 343(ST), 344(ST), 345(ST), 346(IR), 347(IR), 348(CO), 349(ST), 350(HW), 351(HW), 352(SA), 353(SU), 354(HY), 355(PO), 356(CO), 357(HW), 358(HY).

DECEMBER: 359(AT), 360(SM), 361(HY), 362(HY), 363(SM), 364(HY), 365(HY), 366(HY), 367(SU)[c], 368(WW)[c], 369(IR), 370(AT)[c], 371(SM)[c], 372(CO)[c], 373(ST)[c], 374(EM)[c], 375(EM), 376(EM), 377(SA)[c], 378(PO)[c].

VOLUME 80 (1954)

JANUARY: 379(SM)[c], 380(HY), 381(HY), 382(HY), 383(HY), 384(HY)[c], 385(SM), 386(SM), 387(EM), 388(SA), 389(SU)[c], 390(HY), 391(IR)[c], 392(SA), 393(SU), 394(AT), 395(SA)[c], 396(EM)[c], 397(ST)[c].

FEBRUARY: 398(IR)[d], 399(SA)[d], 400(CO)[d], 401(SM)[c], 402(AT)[d], 403(AT)[d], 404(IR)[d], 405(PO)[d], 406(AT)[d], 407(SU)[d], 408(SU)[d], 409(WW)[d], 410(AT)[d], 411(SA)[d], 412(PO)[d], 413(HY)[d].

MARCH: 414(WW)[d], 415(SU)[d], 416(SM)[d], 417(SM)[d], 418(AT)[d], 419(SA)[d], 420(SA)[d], 421(AT)[d], 422(SA)[d], 423(CP)[d], 424(AT)[d], 425(SM)[d], 426(IR)[d], 427(WW)[d].

APRIL: 428(HY)[c], 429(EM)[c], 430(ST), 431(HY), 432(HY), 433(HY), 434(ST).

MAY: 435(SM), 436(CP)[c], 437(HY)[c], 438(HY), 439(HY), 440(ST), 441(ST), 442(SA), 443(SA).

JUNE: 444(SM)[e], 445(SM)[e], 446(ST)[e], 447(ST)[e], 448(ST)[e], 449(ST)[e], 450(ST)[e], 451(ST)[e], 452(SA)[e], 453(SA)[e], 454(SA)[e], 455(SA)[e], 456(SM)[e].

JULY: 457(AT), 458(AT), 459(AT)[c], 460(IR), 461(IR), 462(IR), 463(IR)[c], 464(PO), 465(PO)[c].

AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)[c], 479(HY)[c], 480(ST)[c], 481(SA)[c], 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)[c], 488(ST)[c], 489(HY), 490(HY), 491(HY)[c], 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)[c], 502(WW), 503(WW), 504(WW)[c], 505(CO), 506(CO)[c], 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)[c], 519(IR), 520(IR), 521(IR), 522(IR)[c], 523(AT)[c], 524(SU), 525(SU)[c], 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)[c], 531(EM), 532(EM)[c], 533(PO).

a. Presented at the New York (N.Y.) Convention of the Society in October, 1953.
b. Beginning with "Proceedings-Separate No. 290," published in October, 1953, an automatic distribution of papers was inaugurated, as outlined in "Civil Engineering," June, 1953, page 66.
c. Discussion of several papers, grouped by Divisions.
d. Presented at the Atlanta (Ga.) Convention of the Society in February, 1954.
e. Presented at the Atlantic City (N.J.) Convention in June, 1954.

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